A comparison of algebraic and semi-algebraic proof systems

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Proof systems

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- (correct) $(w, p) \in R \implies w \in L$
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Definition

A proof system Q polynomially simulates R, if for every $(w, p) \in R$ there is $(w, p') \in Q$ such that $p' = \mathrm{poly}(|p|)$.

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- Frege (Schoenfield's system on formulas φ, ψ, η over $\{\vee, \neg\}$):

$$\frac{}{\varphi \vee \neg\varphi} \qquad \frac{\varphi}{\varphi \vee \psi} \qquad \frac{\varphi \vee \varphi}{\varphi} \qquad \frac{\varphi \vee (\psi \vee \eta)}{(\varphi \vee \psi) \vee \eta} \qquad \frac{\varphi \vee \psi \quad \neg\psi \vee \eta}{\varphi \vee \eta}$$

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Any two complete Frege Systems polynomially simulate each other [Reckhow 1975]

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- Extended Frege (additionally abbreviation by fresh variables x):

$$\frac{}{x \leftrightarrow \varphi}$$

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- Systems of polynomial equations over $\mathbb{R}$.
- Polynomials represented as a linear combination of monomials.
- The Boolean axioms $x^2 = x$ are always present.

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\begin{aligned}
x_1 &= 0 &\quad \leftsquigarrow \quad& \overline{x_1} \\
1 - x_2 &= 0 &\quad \leftsquigarrow \quad& x_2 \\
(1 - x_1)x_2(1 - x_3) &= 0 &\quad \leftsquigarrow \quad& x_1 \vee \overline{x_2} \vee x_3
\end{aligned}
$$

$$
\begin{aligned}
\text{for all clauses } C: \quad f_C &= 0 &\quad \leftsquigarrow \quad& C \\
\text{for } i \in [n]: \quad x_i^2 - x_i &= 0 & &
\end{aligned}
$$

Nullstellensatz

A system $f_1(x_1, \dots, x_n) = 0, \dots, f_m(x_1, \dots, x_n) = 0$ of real polynomial equations has no 0/1-solution

$$\iff$$

there are polynomials g_i, q_j such that

$$\sum_{i=1}^{m} g_i f_i + \sum_{j=1}^{n} q_j(x_j^2 - x_j) \; = \; -1.$$

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- The degree of a Nullstellensatz refutation is maximum degree of $g_i f_i$ and $q_j(x_j^2 - x_j)$.
- Refutations of degree d can be found in time $n^{O(d)}$ by solving a system of linear equations.

Sherali-Adams

A system $f_1(x_1, \dots, x_n) = 0, \dots, f_m(x_1, \dots, x_n) = 0$ of real polynomial equations has no 0/1-solution

$$\iff$$

there are polynomials g_i, q_j, p such that

$$\sum_{i=1}^{m} g_i f_i + \sum_{j=1}^{n} q_j(x_j^2 - x_j) + p \;=\; -1,$$

where $p = \sum_{A,B\subseteq[n]} a_{A,B} \cdot \left(\prod_{j\in A} x_j \prod_{j\in B}(1 - x_j)\right)$ with $a_{A,B} \geq 0$.

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- The degree of a Sherali-Adams refutation is maximum degree of $g_i f_i$, $q_j(x_j^2 - x_j)$ and p.
- Refutations of degree d can be found in time $n^{O(d)}$ by solving a linear programme.

Sum-of-squares

A system $f_1(x_1, \dots, x_n) = 0, \ \dots, \ f_m(x_1, \dots, x_n) = 0$ of real polynomial equations has no 0/1-solution

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where $p = \sum_\ell (p_\ell)^2$ is a sum of squared polynomials.

Sum-of-squares

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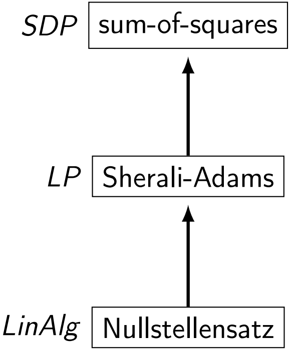
- The degree of a sum-of-squares refutation is maximum degree of $g_i f_i$, $q_j(x_j^2 - x_j)$ and p.
- Refutations of degree d can be found (in time $n^{O(d)}$)$^{*)}$ by solving a semidefinite programme.

$*$) if the bit-length of the coefficients is bounded by $n^{O(d)}$ (not always the case [RW17])

(Semi-)algebraic proof systems

Static systems

$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p = -1$



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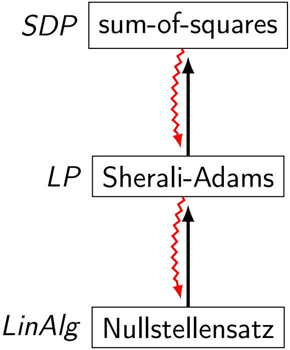

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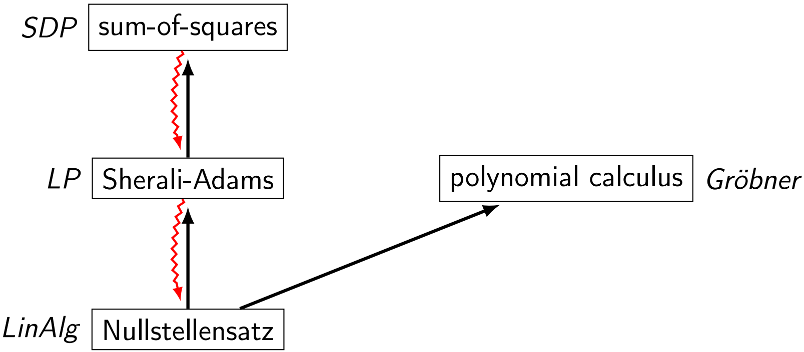
Derivation systems

$\frac{g=0 \qquad f=0}{ag+bf=0}$



Polynomial calculus

Polynomial calculus is a derivation system for polynomials.

$$\frac{}{f_i} \qquad \frac{}{x_j^2 - x_j} \qquad \frac{g \qquad f}{ag + bf} \qquad \frac{f}{x_j f}$$

$f_i = 0$ axiom; x_j variable; f, g, h polynomials; $a, b \in \mathbb{R}$.

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- The degree is the maximum degree of every polynomial in the derivation.
- Refutations of degree d can be found in time $n^{O(d)}$ by a bounded degree variant of the Gröbner Basis Algorithm.
- Extends Nullstellensatz: derive $\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j)$

Polynomial calculus simulates resolution

Resolution (slightly unusual version)

Weakening: $\dfrac{C}{C \vee x}, \quad \dfrac{C}{C \vee \overline{x}}$ Resolution: $\dfrac{C \vee x \qquad C \vee \overline{x}}{C}$

(translating to this special form increases width by at most one and length by a constant factor)

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Observation

Width-d resolution refutation $\Longrightarrow$ degree-d PC refutation.

Reminder: $f_{x_1 \vee \overline{x_2} \vee x_3} = (1 - x_1)x_2(1 - x_3)$

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- Simulation of weakening by multiplication (and lin. comb.):

$$\frac{f_C}{f_C \cdot (1 - x)}, \quad \frac{f_C}{f_C \cdot x}$$

- Simulation of resolution rule by addition:

$$\frac{f_C \cdot x \qquad f_C \cdot (1 - x)}{f_C}$$

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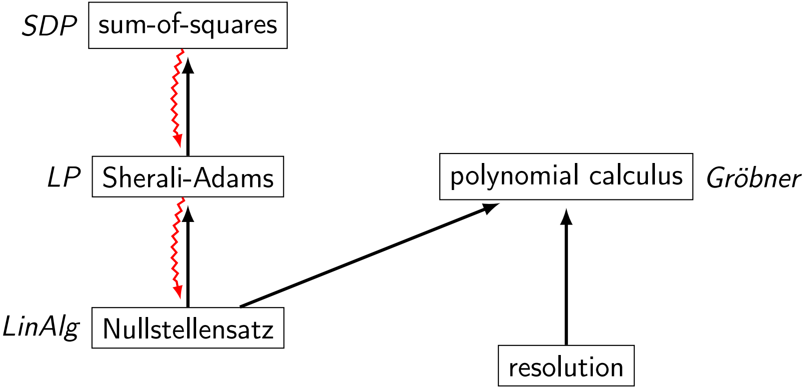

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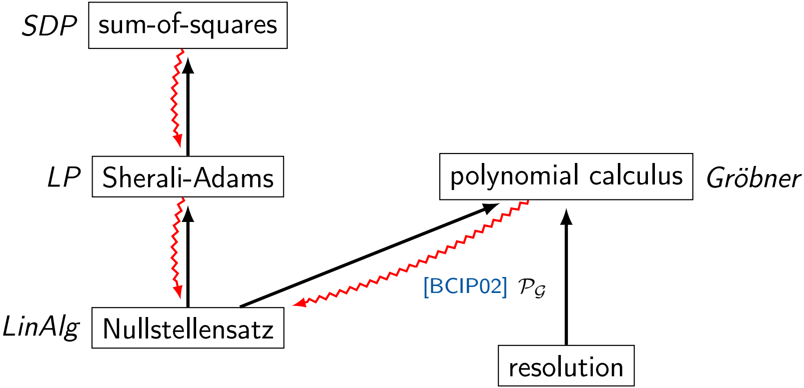

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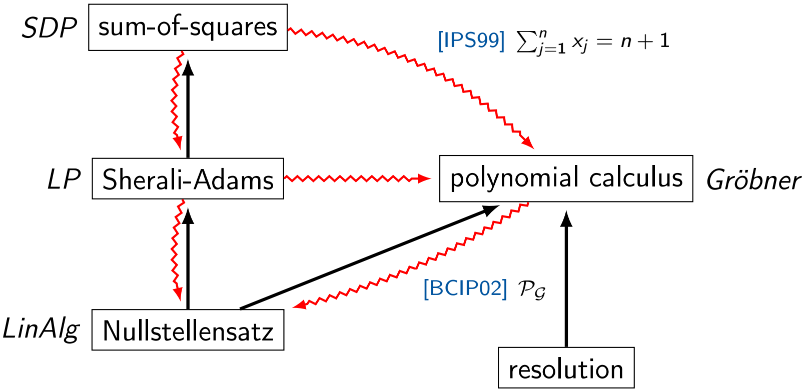

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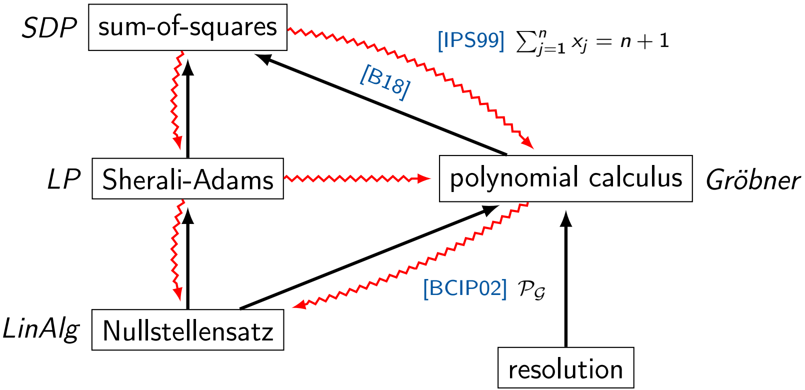

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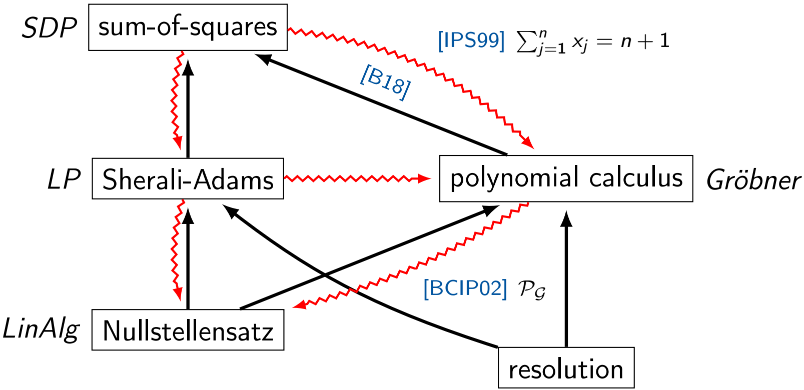

Sherali-Adams simulates resolution

Theorem [DMR09]

If $\Gamma = \{C_1, \ldots, C_m\}$ has a resolution refutation of width d, then $\mathcal{F} = \{f_{C_1} = 0, \ldots, f_{C_m} = 0\}$ has a Sherali-Adams refutation of degree d.

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where $p = \sum_{A,B \subseteq [n]} a_{A,B} \cdot \left(\prod_{j \in A} x_j \prod_{j \in B} (1 - x_j) \right)$ with $a_{A,B} \geq 0$.

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Inductive lemma

If C has a width-d resolution derivation from Γ,
then $-f_C \geq 0$ has a degree-d Sherali-Adams proof from $\mathcal{F}$.

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Proof of the inductive lemma

Resolution rule: $\dfrac{C \vee x \qquad C \vee \overline{x}}{C}$

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$$\sum_i g_i' f_i + \sum_j q_j'(x_j^2 - x_j) + p' = -x \cdot f_C$$
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Sherali-Adams simulates resolution

Proof of the inductive lemma

Resolution rule: $\frac{C \vee x \qquad C \vee \overline{x}}{C}$

$$\sum_i g'_i f_i + \sum_j q'_j(x_j^2 - x_j) + p' = -x \cdot f_C$$
$$\sum_i g''_i f_i + \sum_j q''_j(x_j^2 - x_j) + p'' = -(1 - x) \cdot f_C$$

adding these proofs yields:

$$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p = -f_C$$

Weakening rule: $\frac{C}{C \vee \overline{x}}$ or $\frac{C}{C \vee x}$

$$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p + (1 - x)f_C \quad = -f_C + (1 - x)f_C = -xf_C$$
$$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p + xf_C \quad = -f_C + xf_C = -(1 - x)f_C$$

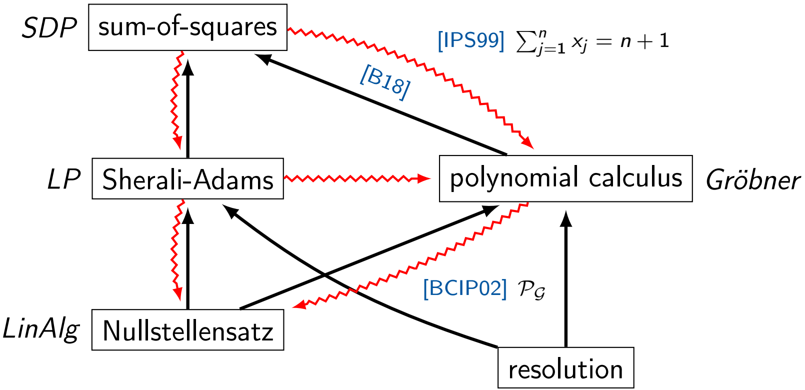
(Semi-)algebraic proof systems

Static systems

$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p = -1$

Derivation systems

$\frac{g=0 \qquad f=0}{ag+bf=0}$



Sum-of-squares simulates polynomial calculus

Theorem [B18]

If $F = \{ f_1 = 0, \ldots, f_m = 0 \}$ has a polynomial calculus refutation of degree d, then it has a sum-of-squares refutation of degree $2d$.

Notation

A sum-of-squares proof of $f \geq 0$ from F is the expression

$$\sum_i g_i f_i + \sum_j q_j (x_j^2 - x_j) + \sum_\ell (p_\ell)^2 = f .$$

Inductive lemma

If f has a degree-d polynomial calculus derivation from F, then $-f^2 \geq 0$ has a degree-$2d$ sum-of-squares proof from F.

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Proof.

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Proof.

Axioms $f = f_i$ and $f = x_j^2 - x_j$ multiplied by $-f$ to derive $-f^2$.

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Proof of the inductive lemma

Inductive lemma
If f has a degree-d polynomial calculus derivation from F,
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Proof.
Linear combination: $\frac{g \qquad h}{ag+bh}$ $\quad f = ag + bh \quad -f^2 = -(ag+bh)^2$

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$$\sum_i g_i' f_i + \sum_j q_j'(x_j^2 - x_j) + \sum_\ell (p_\ell')^2 = -g^2$$
$$\sum_i g_i'' f_i + \sum_j q_j''(x_j^2 - x_j) + \sum_\ell (p_\ell'')^2 = -h^2$$

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(Semi-)algebraic proof systems

Static systems

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Derivation systems

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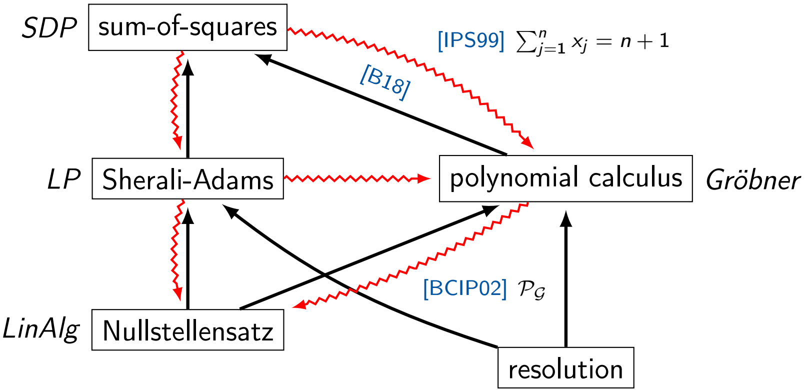

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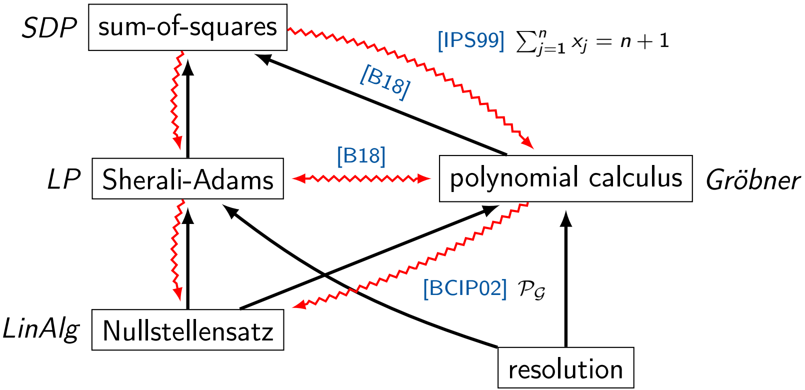

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Suffices to define D on multi-linear monomials $\prod_{i \in I} x_i$.

Nullstellensatz does not simulate resolution & PC

Theorem [BCIP02]

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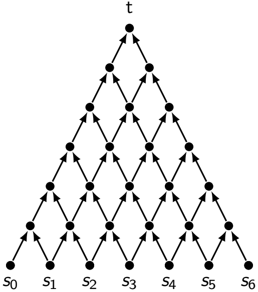

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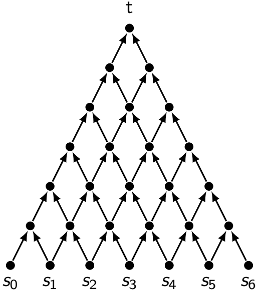

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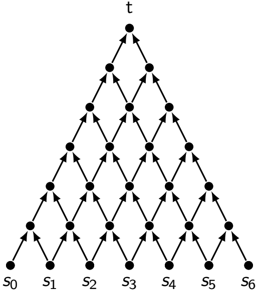

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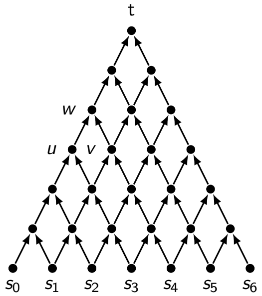

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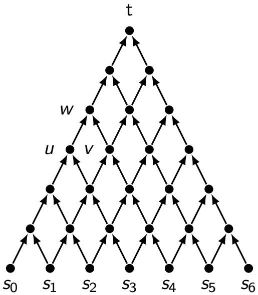
- Resolution refutation of width 3



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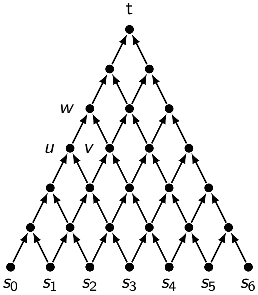

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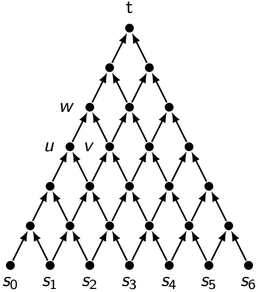

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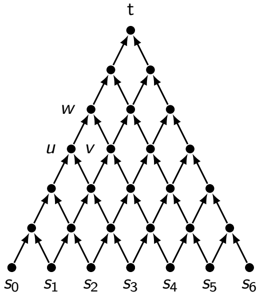

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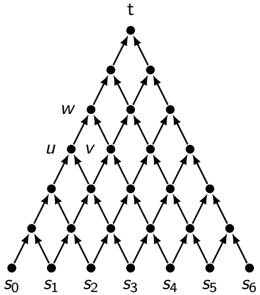

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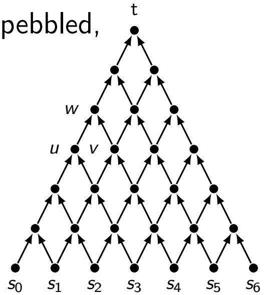

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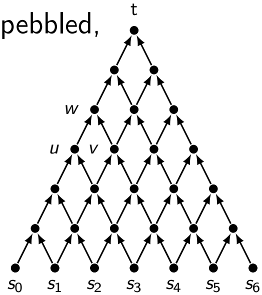

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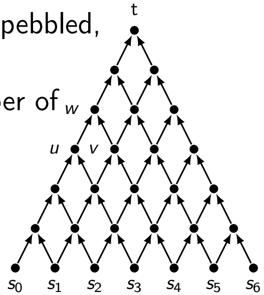

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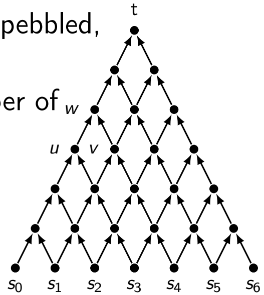

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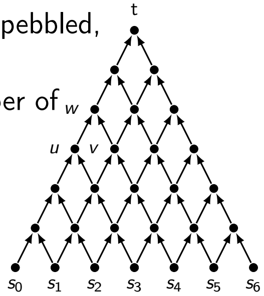

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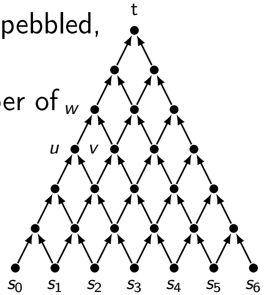

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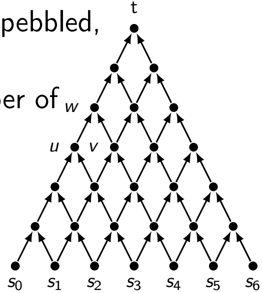

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$\square$

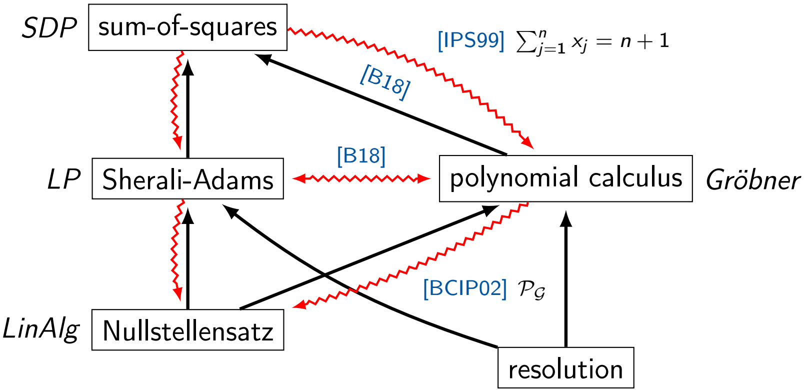
(Semi-)algebraic proof systems

Static systems

$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p = -1$

Derivation systems

$\frac{g=0 \qquad f=0}{ag+bf=0}$



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Observation

Every pebbling contradiction $\mathcal{P}_{\mathcal{G}}$ has a Nullstellensatz refutation of polynomial size (and large degree).

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Let P = Nullstellensatz, Sherali-Adams, or sum-of-squares.
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Proof. For every x uniformly at random set either x^0 or x^1 to 0.
If there are at most 2^{d-1} multi-linear monomials of degree $\geq d$, they all vanish with non-zero probability,
leading to a P-refutation of $\mathcal{F}$ of degree $< d$. $\square$

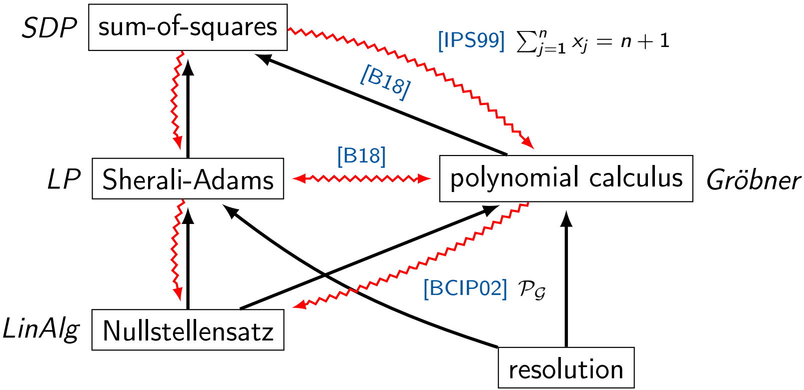
(Semi-)algebraic proof systems

Static systems

$\sum_i g_i f_i + \sum_j q_j(x_j^2 - x_j) + p = -1$

Derivation systems

$\frac{g=0 \qquad f=0}{ag+bf=0}$



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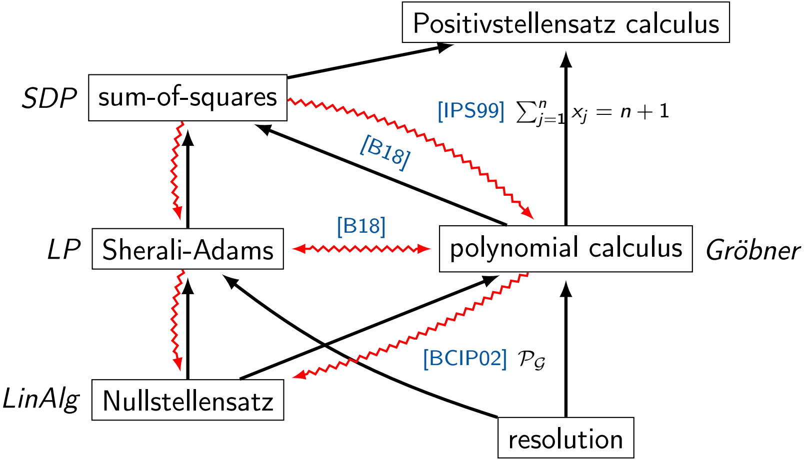

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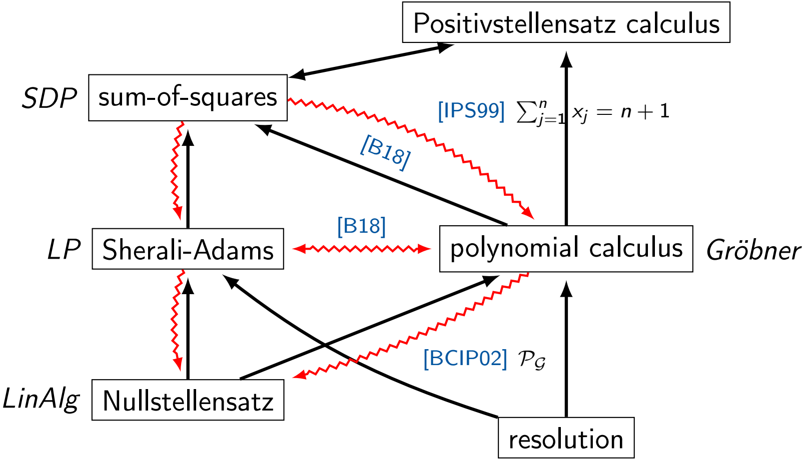

Positivstellensatz and Positivstellensatz calculus

Let $\mathcal{F} = \{f_1 = 0, \ldots, f_m = 0\}$ and $\mathcal{H} = \{h_1 \geq 0, \ldots, h_s \geq 0\}$.

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A Positivstellensatz proof of $f \geq 0$ from $(\mathcal{F}, \mathcal{H})$ is

$$\sum_{i=1}^{m} g_i f_i + \sum_{j=1}^{n} q_j(x_j^2 - x_j) + p + \sum_{I \subseteq [s]} p_I \prod_{\ell \in I} h_\ell = f,$$

where p, p_I are sums-of-squares.

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A Positivstellensatz calculus proof of $f \geq 0$ from $(\mathcal{F}, \mathcal{H})$ is a polynomial calculus proof of

$$f - p - \sum_{I \subseteq [s]} p_I \prod_{\ell \in I} h_\ell \qquad \text{from } \mathcal{F}.$$

Positivstellensatz vs. Positivstellensatz calculus

Theorem [B18]

Positivstellensatz $\equiv$ Positivstellensatz calculus on Boolean systems.

Proof. $(\mathcal{F}, \mathcal{H})$ has a Positivstellensatz calculus refutation

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$\iff -1 - p - \sum_{I \subseteq [s]} p_I \prod_{\ell \in I} h_\ell$ has a PC derivation from $\mathcal{F}$

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$\implies$ (ind. lemma) there is a degree-$2d$ SOS proof
$\sum_{i=1}^m g_i f_i + \sum_{j=1}^n q_j (x_j^2 - x_j) + p' = -(-1 - p - \sum_{I \subseteq [s]} p_I \prod_{\ell \in I} h_\ell)^2.$

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Interestingly, on non-Boolean systems this is not the case:
$\mathcal{F}_n^{\text{ts}} := \{yx_1 = 1,\ x_1^2 = x_2,\ x_2^2 = x_3,\ \ldots,\ x_{n-1}^2 = x_n,\ x_n = 0\}$

Theorem [GV01]

(without $x^2 - x = 0$ axioms:)

- $\mathcal{F}_n^{\text{ts}}$ has Positivstellensatz calculus refutations of degree $O(n)$.
- $\mathcal{F}_n^{\text{ts}}$ requires Positivstellensatz refutations of degree $2^{\Omega(n)}$.

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